

# Convergence Analysis of the Hynomics Incremental Optimizer

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## Abstract

In this note we analyze a perturbation-based optimization algorithm for the implementation of real time incremental repair feedback systems. This algorithm is a central element in the implementation of real time reactive planning scheduling and execution applications that use the Hynomics agent architecture.

## 1 Introduction

In this note we describe a second order optimization algorithm for the implementation of real time incremental repair feedback systems. This algorithm is the central element in the deployment of real time reactive planning, scheduling and execution applications.

The class of optimization problems we are interested is formulated as follows:

$$\min_{\mathbf{u}} \int_0^T L(\tilde{\mathbf{x}}(t), \mathbf{u}(t)) dt + \Phi(\tilde{\mathbf{x}}(T), T) \quad (1)$$

subject to the following constraints:

$$\dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbf{f}}(\tilde{\mathbf{x}}(t), \mathbf{u}(t), \varphi(t)) \quad (2)$$

$$\tilde{\mathbf{x}}(0) = \mathbf{x}_0 \quad (3)$$

Where  $\tilde{\mathbf{x}}(t) \in \mathcal{R}^n$ , will be referred to as the state of the system,  $\mathbf{u}(t) \in \mathcal{U} \subseteq \mathcal{R}^m$ ,  $m \leq n$ , will be referred to as the control,  $\boldsymbol{\varphi}(t) \in \mathcal{R}^n$  is a continuous function called the disturbance,  $t \in [0, T]$  is the time variable, and  $L, \mathbf{f}$  are piecewise  $C^2$  functions in both of their arguments with  $L(\mathbf{z}, \mathbf{w}) > 0$  for  $\mathbf{z}, \mathbf{w} \neq \mathbf{0}$ .  $L$ , and  $\tilde{\mathbf{f}}$  are called the running criterion and the dynamics functions respectively. Finally  $\mathcal{U}$  is a compact subset.

## 2 Reformulation

We want to reformulate the problem defined by 1-3 as a terminal optimization problem. The purpose of this reformulation is to simplify subsequent procedures for the computation of the incremental functions of the control and state of the system around a nominal state and control functions:  $\tilde{x}^0(t), u^0(t)$ .

We define a variable,  $x_{n+1}(t)$  as a solution of the following differential equation:

$$\dot{x}_{n+1}(t) = L(\tilde{\mathbf{x}}(t), \mathbf{u}(t)) \quad (4)$$

with initial condition,  $x(0) = 0$ . Then, problem(1)- (3) takes the form:

$$\min_u \Omega(\mathbf{x}(T), T) \quad (5)$$

subject to:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\varphi}(t)) \quad (6)$$

Where,

$$\mathbf{x}(t) = \begin{bmatrix} \tilde{\mathbf{x}}(t) \\ x_{n+1}(t) \end{bmatrix} \quad (7)$$

is the augmented state,

$$\Omega(\mathbf{x}(T), T) = x_{n+1}(T) + \Phi(\tilde{\mathbf{x}}(T), T) \quad (8)$$

is the terminal criterion and

$$\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\varphi}(t)) = \begin{bmatrix} \tilde{\mathbf{f}}(\tilde{\mathbf{x}}(t), \mathbf{u}(t), \boldsymbol{\varphi}(t)) \\ L(\tilde{\mathbf{x}}(t), \mathbf{u}(t)) \end{bmatrix} \quad (9)$$

are the augmented dynamics. The initial conditions are given by:

$$\mathbf{x}(0) = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{0} \end{bmatrix} \quad (10)$$

The set of expressions (5)-(10) provide an equivalent formulation of problem (1)-(3), which is better suited for our subsequent computations.

### 3 Necessary conditions for optimality

Pontryagin's Minimum Principle [[1]] provides a set of necessary conditions for the problem (5)-(10). These conditions are expressed in terms of the Hamiltonian function of the system given by:

$$H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\varphi}(t), \boldsymbol{\lambda}(t)) = \boldsymbol{\lambda}^T(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\varphi}(t)) \quad (11)$$

where  $\boldsymbol{\lambda}(t) \in \mathcal{R}^{n+1}$  is a continuous function called the costate of the system. In terms of (11), the minimum principle is stated as follows:

Let  $\mathbf{u}^*(t) \in \mathcal{U}, t \in [0, T)$  be an optimal solution of (5)-(10) with  $\mathbf{x}^*(t), t \in [0, T)$  the corresponding state. Then,

$$H(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\varphi}(t), \boldsymbol{\lambda}(t)) \leq H(\mathbf{x}^*(t), \mathbf{u}(t), \boldsymbol{\varphi}(t), \boldsymbol{\lambda}(t)) \quad (12)$$

For all measurable, continuous  $u, u(t) \in \mathcal{U} t \in [0, T)$

$$\dot{\boldsymbol{\lambda}}(t) = -\mathbf{H}_{\boldsymbol{x}}^T(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\varphi}(t), \boldsymbol{\lambda}(t))$$

$$\dot{\mathbf{x}}^*(t) = \mathbf{H}_{\boldsymbol{\lambda}}^T(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\varphi}(t), \boldsymbol{\lambda}(t))$$

$$\boldsymbol{\lambda}(T) = \boldsymbol{\Omega}_x(x(T), T)$$

$$\mathbf{x}^*(0) = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{0} \end{bmatrix}$$

are necessary conditions for optimality of (5)-(10). If  $H$  is continuously twice differentiable with respect to  $\mathbf{u}$  in the vicinity of  $u^*$ , the first condition in (12) can be replaced by

$$\mathbf{H}_{\mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\varphi}(t), \boldsymbol{\lambda}(t)) = \mathbf{0} \quad (13)$$

and

$$\mathbf{H}_{\mathbf{u}\mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\varphi}(t), \boldsymbol{\lambda}(t)) \text{ is positive semidefinite} \quad (14)$$

The purpose of this note is to develop an algorithm for computing an approximation to

$\mathbf{u}^*(t), t \in [0, T)$  that satisfies the sufficient condition

$$\mathbf{H}_{\mathbf{u}\mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\varphi}(t), \boldsymbol{\lambda}(t)) \text{ is positive definite} \quad (15)$$

We will formulate this algorithm in the next section and study some its convergence properties in 5.

## 4 Incremental Optimizer Procedure

In terms of the Hamiltonian function 11, we can express the problem (5)-(10) as an unconstrained optimization problem as follows:

$$\min_u \int_0^T \left\{ H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\varphi}(t), \boldsymbol{\lambda}(t)) - \lambda^T(t) \dot{\mathbf{x}}(t) \right\} dt + \Omega(\mathbf{x}(T), T) \quad (16)$$

Integrating by parts the second term under the integral, (16) takes the form:

$$\min_u \int_0^T \left\{ H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\varphi}(t), \boldsymbol{\lambda}(t)) + \dot{\lambda}^T(t) \mathbf{x}(t) \right\} dt + \Omega(\mathbf{x}(T), T) - \boldsymbol{\lambda}^T(T) \mathbf{x}(T) \quad (17)$$

Let us suppose that a suitable approximation  $\mathbf{u}^k$  to  $\mathbf{u}^*$  is available and let  $\mathbf{x}^k$  the corresponding state trajectory that satisfies (6)

$$\dot{\mathbf{x}}^k(t) = \mathbf{f}(\mathbf{x}^k(t), \mathbf{u}^k(t), \boldsymbol{\varphi}(t)) \quad (18)$$

with initial conditions:

$$\mathbf{x}^k(0) = \mathbf{x}(0) \quad (19)$$

We want to find the  $k$ th optimal increment  $\Delta \mathbf{u}^k(t)$  that solves the following optimization problem:

$$\min_{\Delta \mathbf{u}^k} \int_0^T \left\{ \begin{array}{l} H(\mathbf{x}^k(t) + \Delta \mathbf{x}^k(t), \mathbf{u}^{k+1}(t), \boldsymbol{\varphi}(t), \boldsymbol{\lambda}^k(t)) + \\ \left( \dot{\boldsymbol{\lambda}}^k(t) \right)^T \left( \mathbf{x}^k(t) + \Delta \mathbf{x}^k(t) \right) + \end{array} \right\} dt \quad (20)$$

$$+ \Omega(\mathbf{x}^k(T) + \Delta \mathbf{x}^k(T), T) - \left( \boldsymbol{\lambda}^k \right)^T(T) \left( \mathbf{x}^k(T) + \Delta \mathbf{x}^k(T) \right)$$

with

$$\mathbf{u}^{k+1}(t) = \mathbf{u}^k(t) + \Delta \mathbf{u}^k(t) \quad (21)$$

for each  $t$ . The perturbation term  $\Delta \mathbf{x}^k(t)$  is computed as a solution of a linear differential equation that we will describe later in this section (equation (25)).

We choose  $\boldsymbol{\lambda}^k(t)$  in (20) so that it satisfies the following differential equation

$$\left( \dot{\boldsymbol{\lambda}}^k(t) \right)^T = -\mathbf{H}_{\mathbf{x}}(\mathbf{x}^k(t), \mathbf{u}^k(t), \boldsymbol{\varphi}(t), \boldsymbol{\lambda}^k(t)) \quad (22)$$

with boundary conditions:

$$\left( \boldsymbol{\lambda}^k(T) \right)^T = \boldsymbol{\Omega}_{\mathbf{x}}(\mathbf{x}^k(T), T) \quad (23)$$

We note that by selecting  $\mathbf{x}^k(t), \boldsymbol{\lambda}^k(t), t \in [0, T]$  so hat they satisfy (18) and (22) respectively, with boundary conditions, (19) and (23) our approximation triple  $(\mathbf{x}^k(t), \mathbf{u}^k(t), \boldsymbol{\lambda}^k(t), t \in [0, T])$  automatically satisfies all but the first necessary condition of the minimum principle (see (12)).

By definition, the Hamiltonian  $H$  and the terminal function  $\Omega$  are twice continuously differentiable in each of its arguments. Therefore we can approximate (20) with

$$\begin{aligned} \min_{\Delta \mathbf{u}^k} \int_0^T \left\{ \begin{aligned} & H|{}^k + \left( \mathbf{H}_{\mathbf{x}}|{}^k + \left( \dot{\boldsymbol{\lambda}}^k(t) \right)^T \right) \Delta \mathbf{x}^k(t) + \mathbf{H}_{\mathbf{u}}|{}^k \Delta \mathbf{u}_k + \\ & \frac{1}{2} \begin{bmatrix} \left( \Delta \mathbf{x}^k(t) \right)^T & \left( \Delta \mathbf{u}^k(t) \right)^T \end{bmatrix} \begin{bmatrix} \mathbf{H}_{\mathbf{xx}}|{}^k & \mathbf{H}_{\mathbf{xu}}|{}^k \\ \mathbf{H}_{\mathbf{ux}}|{}^k & \mathbf{H}_{\mathbf{uu}}|{}^k \end{bmatrix} \begin{bmatrix} \Delta \mathbf{x}^k(t) \\ \Delta \mathbf{u}^k(t) \end{bmatrix} \end{aligned} \right\} dt \\ + \Omega(\mathbf{x}^k(T), T) + \left( \boldsymbol{\Omega}_{\mathbf{x}}(\mathbf{x}^k(T), T) - \left( \boldsymbol{\lambda}^k(T) \right)^T \right) \Delta \mathbf{x}^k(T) + \\ \frac{1}{2} \left( \Delta \mathbf{x}^k(T) \right)^T \boldsymbol{\Omega}_{\mathbf{xx}}(\mathbf{x}^k(T), T) \Delta \mathbf{x}^k(T) + \text{Higher-order-terms} \quad (24) \end{aligned}$$

In (24) the notation  $|{}^k$  denotes the argument  $(\mathbf{x}^k(t), \mathbf{u}^k(t), \boldsymbol{\varphi}(t), \boldsymbol{\lambda}^k(t))$ .

The incremental functions  $\Delta \mathbf{x}^k, \Delta \mathbf{u}^k$  are not independent; they are related to each other by a linear differential equation, that we alluded to above, for the incremental state  $\Delta \mathbf{x}^k$ . This equation is obtained by perturbing to first order the equation in 6 and retaining only first order terms:

$$\Delta \dot{\mathbf{x}}^k(t) = \mathbf{f}_{\mathbf{x}}(\mathbf{x}^k(t), \mathbf{u}^k(t), \boldsymbol{\varphi}(t)) \Delta \mathbf{x}^k(t) + \mathbf{f}_{\mathbf{u}}(\mathbf{x}^k(t), \mathbf{u}^k(t), \boldsymbol{\varphi}(t)) \Delta \mathbf{u}^k(t) \quad (25)$$

with initial conditions,

$$\Delta \mathbf{x}^k(0) = \mathbf{0}$$

Equations (24) and (25) state our incremental optimization procedure.

We now simplify the criterion in (24). Using (22) and (23) in(24) we obtain:

$$\begin{aligned} \min_{\Delta \mathbf{u}^k} \int_0^T \left\{ \begin{aligned} & H|{}^k + \mathbf{H}_{\mathbf{u}}|{}^k \Delta \mathbf{u}_k + \\ & \frac{1}{2} \begin{bmatrix} \left( \Delta \mathbf{x}^k(t) \right)^T & \left( \Delta \mathbf{u}^k(t) \right)^T \end{bmatrix} \begin{bmatrix} \mathbf{H}_{\mathbf{xx}}|{}^k & \mathbf{H}_{\mathbf{xu}}|{}^k \\ \mathbf{H}_{\mathbf{ux}}|{}^k & \mathbf{H}_{\mathbf{uu}}|{}^k \end{bmatrix} \begin{bmatrix} \Delta \mathbf{x}^k(t) \\ \Delta \mathbf{u}^k(t) \end{bmatrix} \end{aligned} \right\} dt \\ + \Omega(\mathbf{x}^k(T), T) + \frac{1}{2} \left( \Delta \mathbf{x}^k(T) \right)^T \boldsymbol{\Omega}_{\mathbf{xx}}(\mathbf{x}^k(T), T) \Delta \mathbf{x}^k(T) \quad (26) \end{aligned}$$

Further, since the term  $H|{}^k$  in (26) has no explicit dependency on either  $\Delta \mathbf{x}^k$  or  $\Delta \mathbf{u}^k$ ; it may be dropped from the optimization criterion. This it also true for the term  $\Omega(\mathbf{x}^k(T), T)$ . Thus the incremental optimization problem is given by:

$$\min_{\Delta \mathbf{u}^k} \int_0^T \left\{ \frac{1}{2} \begin{bmatrix} (\Delta \mathbf{x}^k(t))^T & (\Delta \mathbf{u}^k(t))^T \end{bmatrix} \begin{bmatrix} \mathbf{H}_{\mathbf{u}}|{}^k \Delta \mathbf{u}^k(t) + \\ \mathbf{H}_{\mathbf{xx}}|{}^k & \mathbf{H}_{\mathbf{xu}}|{}^k \\ \mathbf{H}_{\mathbf{ux}}|{}^k & \mathbf{H}_{\mathbf{uu}}|{}^k \end{bmatrix} \begin{bmatrix} \Delta \mathbf{x}^k(t) \\ \Delta \mathbf{u}^k(t) \end{bmatrix} \right\} dt \quad (27)$$

$$+ \frac{1}{2} (\Delta \mathbf{x}^k(T))^T \Omega_{\mathbf{xx}}(\mathbf{x}^k(T), T) \Delta \mathbf{x}^k(T)$$

Subject to constraint (25). This is a linear quadratic-affine optimization problem. We will write down the feedback solution for this problem. The derivation can be found in [[2]].

The incremental control  $\Delta \mathbf{u}^k$  optimizing (27) is given by the following affine feedback expression [2]:

$$\Delta \mathbf{u}^k(t) = -\mathbf{H}_{\mathbf{uu}}^{-1}|{}^k [\mathbf{H}_{\mathbf{ux}}|{}^k + \mathbf{f}_{\mathbf{u}}^T|{}^k \Sigma^k(t)] \Delta \mathbf{x}^k(t) - \mathbf{H}_{\mathbf{uu}}^{-1}|{}^k [\mathbf{f}_{\mathbf{u}}^T|{}^k \mathbf{g}^k(t) + \mathbf{H}_{\mathbf{u}}^T|{}^k] \quad (28)$$

In (28)  $\Sigma(t)$  is an  $\mathcal{R}^{(n+1)^2}$ -valued matrix function satisfying the following differential equation:

$$\begin{aligned} \dot{\Sigma}^k(t) = & -\Sigma^k(t) [\mathbf{f}_{\mathbf{x}}|{}^k - \mathbf{f}_{\mathbf{u}}|{}^k \mathbf{H}_{\mathbf{uu}}^{-1}|{}^k \mathbf{H}_{\mathbf{ux}}|{}^k] - [\mathbf{f}_{\mathbf{x}}^T|{}^k - \mathbf{H}_{\mathbf{xu}}|{}^k \mathbf{H}_{\mathbf{uu}}^{-1}|{}^k \mathbf{f}_{\mathbf{u}}^T|{}^k] \Sigma^k(t) \\ & + \Sigma^k(t) \mathbf{f}_{\mathbf{u}}|{}^k \mathbf{H}_{\mathbf{uu}}^{-1}|{}^k \mathbf{f}_{\mathbf{u}}^T|{}^k \Sigma^k(t) \\ & - [\mathbf{H}_{\mathbf{xx}}|{}^k - \mathbf{H}_{\mathbf{xu}}|{}^k \mathbf{H}_{\mathbf{uu}}|{}^k \mathbf{H}_{\mathbf{ux}}|{}^k] \end{aligned} \quad (29)$$

Under general controllability assumptions, [[3]], [[4]] for each  $t$ ,  $\Sigma^k(t)$  is symmetric and positive definite. Equation (29) satisfies the following boundary condition:

$$\Sigma^k(T) = \Omega_{\mathbf{xx}}(\mathbf{x}^k(T), T) \quad (30)$$

In (28),  $\mathbf{g}^k(t)$  is an  $\mathcal{R}^{(n+1)}$ -valued function satisfying the following differential equation:

$$\begin{aligned} \dot{\mathbf{g}}^k(t) = & [-\mathbf{f}_{\mathbf{x}}^T|{}^k + \Sigma(t) \mathbf{f}_{\mathbf{u}}|{}^k \mathbf{H}_{\mathbf{uu}}^{-1}|{}^k \mathbf{f}_{\mathbf{u}}^T|{}^k + \mathbf{H}_{\mathbf{xu}}|{}^k \mathbf{H}_{\mathbf{uu}}^{-1}|{}^k \mathbf{f}_{\mathbf{u}}^T|{}^k] \mathbf{g}^k(t) \\ & + [\Sigma(t) \mathbf{f}_{\mathbf{u}}|{}^k \mathbf{H}_{\mathbf{uu}}^{-1}|{}^k + \mathbf{H}_{\mathbf{xu}}|{}^k \mathbf{H}_{\mathbf{uu}}^{-1}|{}^k] \mathbf{H}_{\mathbf{u}}^T|{}^k \end{aligned} \quad (31)$$

with boundary condition

$$\mathbf{g}^k(T) = 0$$

We summarize the proposed incremental procedure in the flowchart of figure 1.

The flowchart of figure 1, is composed of two sequential sub-subprocedures: A global subprocedure and an incremental subprocedure. The global sub procedure generates state and costate trajectory iterates  $\mathbf{x}^k, \boldsymbol{\lambda}^k$  given the current

control trajectory  $\mathbf{u}^k$  and the disturbance trajectory  $\varphi$ . The incremental sub-procedure generates a control increment trajectory  $\Delta\mathbf{u}^k$  so that, as we will show in the next section, the resulting control  $\mathbf{u}^{k+1} = \mathbf{u}^k + \Delta\mathbf{u}^k$  trajectory is closer to the optimal  $\mathbf{u}^*$ .

## 5 Convergence Analysis

We start with The convergence theorem we wish to proof.

**Theorem 1** *Let  $H(\mathbf{x}(t), \mathbf{u}(t), \varphi(t), \boldsymbol{\lambda}(t))$  be defined as in (11). Assume that  $H_{\mathbf{u}\mathbf{u}}(\mathbf{x}^k(t), \mathbf{u}^k(t), \varphi(t), \boldsymbol{\lambda}^k(t))$  is positive definite along a tubular neighborhood of  $(\mathbf{x}^k(t), \mathbf{u}^k(t), \boldsymbol{\lambda}^k(t), t \in [0, T])$  where the sequences  $\mathbf{x}^k(t), \mathbf{u}^k(t), \boldsymbol{\lambda}^k(t)$  are computed as follows:  $\mathbf{x}^k(t)$  is the solution of (18), (19),  $\boldsymbol{\lambda}^k(t)$  is the solution of (22), (23) and  $\mathbf{u}^k(t)$  is computed by (21) then*

$$H|^{k+1} < H|^{k} \quad (32)$$

**Proof.** Expanding  $H|^{k+1}$  around  $|^k$  and retaining up to second order terms, we obtain,

$$\begin{aligned} H|^{k+1} &= H|^{k} + H|_x^k \Delta x|^{k} + H|_u^k \Delta u|^{k} \\ &\quad + \frac{1}{2} [\Delta x^T|^{k} H|_{xx}^k \Delta x|^{k} + \Delta x^T|^{k} H|_{xu}^k \Delta u|^{k} + \Delta u^T|^{k} H|_{uu}^k \Delta u|^{k}] \\ &\quad + O\left((\Delta x|^{k})^2, (\Delta u|^{k})^2\right) \end{aligned} \quad (33)$$

From (28), and using a result from [5, eq 231, section 62], theorem from section 6.2, we obtain the following expression

$$H|^{k+1} = H|^{k} - \Delta x^T|^{k} Q|_0^k \Delta x|^{k} - \Delta x^T|^{k} Q|_1^k - \psi|^{k}(t), \quad (34)$$

where  $Q|_0^k$  is a positive definite symmetric matrix,  $Q|_{1j}^k$  is a positive real, and  $\psi|^{k}(t)$  is a non-negative function of time.

Thus from (33) and (34), (32) follows. ■

The following corollary establishes the convergence results we wish to prove.

**Corollary 2**

$$H|^{k} \rightarrow H^* \quad (35)$$

**Proof.** By construction, (11),  $H$  is a non-negative function bounded below and, by Theorem 1,  $H|^{k}$  is a decreasing sequence of non-negative real numbers. Therefore it is a Cauchy sequence that convergent to a non-negative real number  $H^*$  for each time  $t$ . We claim that  $H^*(t) = H(x^*(t), u^*(t), \varphi(t), \lambda(t))$ . To see this, notice that  $\lambda|^{k}(t)$  satisfies (22) and (23) for each  $k$ . By the convergence theorem of ODE's,

$$\dot{\lambda}(t) = -H_x(x^*(t), u^*(t), \lambda(t), \varphi(t)). \quad (36)$$

Similarly, by (18),

$$\dot{x}^*(t) = f(x^*(t), u^*(t), \varphi(t)). \quad (37)$$

Finally, since  $[-\mathbf{f}_x^T]^k + \Sigma(t)\mathbf{f}_u|^k\mathbf{H}_{uu}^{-1}|^k\mathbf{f}_u^T|^k + \mathbf{H}_{xu}|^k\mathbf{H}_{uu}^{-1}|^k\mathbf{f}_u^T|^k$  is a stability matrix [6],  $g|^k \rightarrow 0$ . Therefore  $\Delta u|^k \rightarrow 0$ ,  $u|^k \rightarrow u^*$ . This implies

$$H_u(x^*(t), u^*(t), \lambda(t), \varphi(t)) \rightarrow 0.$$

Then,

$$H^*(t) = H(x^*(t), u^*(t), \lambda(t), \varphi(t)).$$

■

In summary, the proposed algorithmic schema of Figure 1 converges uniformly to the optimal solution. Moreover, it can be shown, [7], [2], that if the dynamics is bilinear and the criterion is bilinear affine, the system converges to the optimum from any feasible solution in one step.

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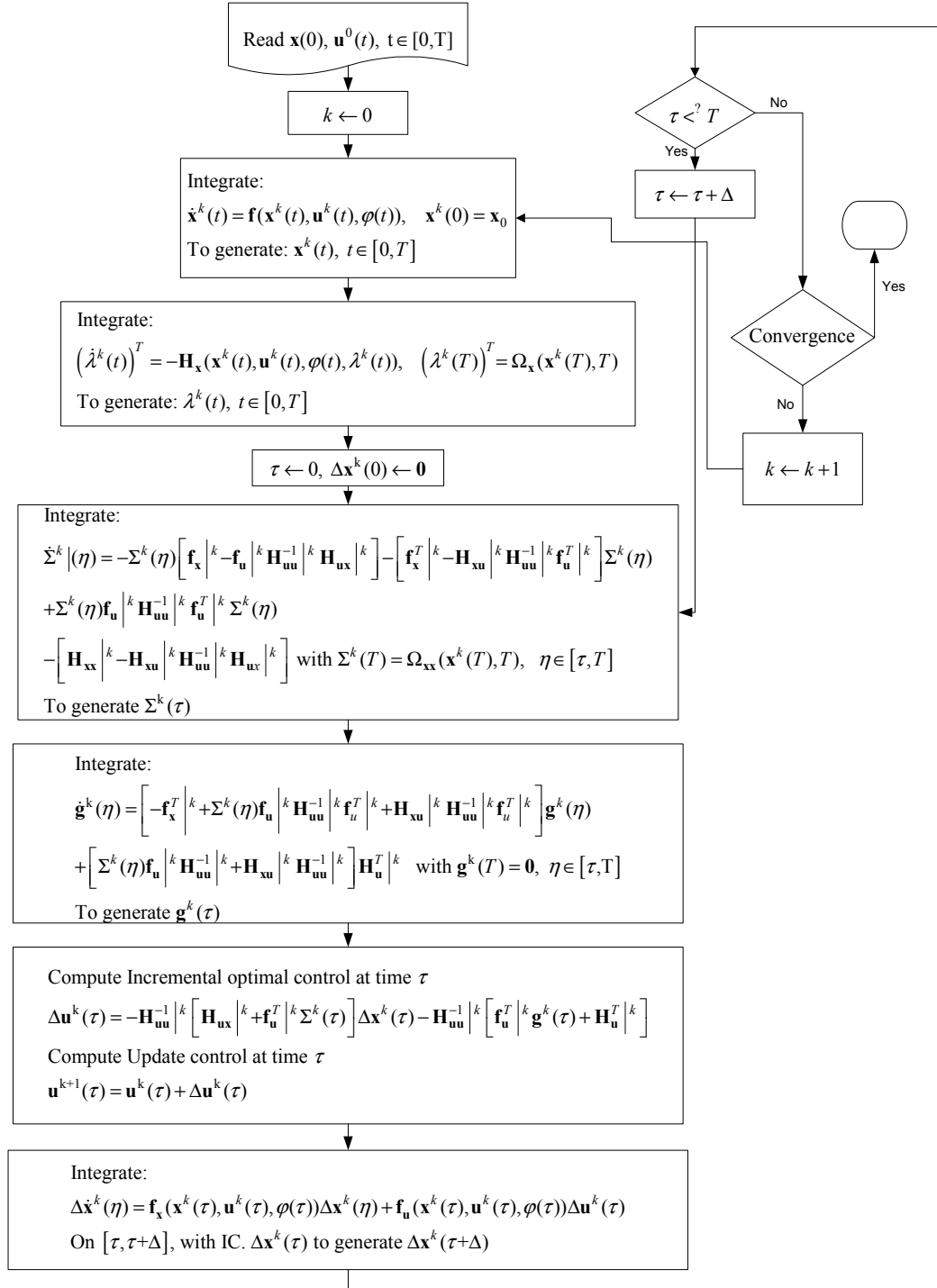


Figure 1: Incremental optimizer: conceptual flowchart