Multi-Echelon Inventory Planning System II: Continualization

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Abstract

This is the second of two papers in which we describe a hybrid systems approach to supply chain management problems. We provide an outline of a design procedure of a Multiple Agent Hybrid Control Architecture (MAHCA) network for the deployment of a reactive, agent based planner for a single product multi-echelon inventory system. The objective of the system is to generate a near optimal distributed stocking plan policy. This policy approximates a centralized stocking policy which minimizes the expected value of a suitably defined cost functional. In the first paper [8], we described the basic model for any agent that controls a single node in the network. In this paper, we shall describe a continualization procedure which will reduce the agent optimization problem presented in [8] to a more standard non-linear optimal control problem that has the appropriate form to be solved by a MAHCA system.

1. Introduction

This is the second of two papers in which we describe a hybrid systems approach to supply chain management problems. In these two papers, we provide an outline of a design procedure of a Multiple Agent Hybrid Control Architecture (MAHCA) network for the deployment of a reactive, agent based planner for a single product, multi-echelon inventory system. The objective of the system is to generate a near optimal distributed stocking plan policy. We will generate a policy that approximates a centralized stocking policy which minimizes the expected value of a suitably defined cost functional. In the first paper [8], we described the basic model of an agent controlling a single node in the distribution network plus a general criterion for the whole network based on a rule based synchronization procedure. In this paper, we shall describe a continualization procedure which will reduce our problem to a more standard non-linear optimal control problem that has the appropriate form to be solved by a MAHCA system.

We shall model a distribution network for a single product p as a directed graph $G = (V, A)$. Each vertex j in the set V represents a distribution center (DC). The existence of an edge (i, j) implies that the DC represented by i acts a source of supply for the DC represented by j. When a DC can receive supply from multiple sources, we assume that there is a prespecified order in which the sources will be polled for supply. The transportation time between any two DC's i and j with $(i, j) \in A$ is one time period. We allow one infinite supply source that directly supplies several nodes in the network. We assume that there is a transportation lead-time of $l_j \geq 1$ time periods from the infinite source external supplier to each node j directly connected to the infinite sources. We assume that the demand for p at vertex v in time period t follow a stochastic process whose cumulative density function (c.d.f.) is given by the function $F_i^t(\cdot)$. For any vertex i, we assume that the demand for p at vertex i is greater than zero in every period and that the demands for any two nodes are independent. We assume that the cost of holding inventory at DC i is h_i dollars per unit for each time period and that the penalty for not satisfying demand is s_i dollars per unit for each time period.

Next we list the model variables for a generic distribution center termed the i-th distribution center where i is the label of the node representing the center in the graph of the system.
(1) **Inventory (state variable)**: $x_i(t)$ is the number of units of the product carried over from the previous period at node DC i at the beginning of period t. We assume $x_i(t) \geq 0\ \forall i, t$.
(2) **Ordered amount(decision variable)** $u_{ij}(t)$ is the number of units of the product ordered by DC i from DC $j \in U_i$ at the end of period t where U_i is the index set of the upstream nodes directly connected to node DC i. We assume $u_{ij}(t) \geq 0\ \forall i, j, t$.
(3) **Sent amount (decision variable)** $v_{ij}(t)$ is the number of units of the product sent from DC i to DC $j \in V_i$ at the end of period t where V_i is the index set

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of the downstream nodes adjacent to node DC i. We assume $0 \leq v_{ij}(t) \leq u_{ji}(t)$, $\forall i, t \forall j \in V_i$.
(4) **External demand (auxilary variable)** $y_i(t)$ is the number of units of product satisfaction delivered by DC i to meet the external demand $d_i(t)$. We assume $0 \leq y_i(t) \leq d_i(t)$ $\forall i, t$.
(5) **Total demand (auxiliary variable)** $D_i(t) = d_i(t) + \sum_{j \in V_j} u_{ji}(t-1)$, $\forall i$.
(6) **Total availability (auxilary variable)** $S_i(t) = x_i(t) + \sum_{j \in U_i} v_{ji}(t - \omega_{ji}$, $\forall i > 0$ and $S_0(t) = \infty$. Here ω_{ij} is the transportation lead-time from DC j to DC i. Thus for $j > 0, i = 3, \ldots, 6$, $\omega_{ji} = 1$ and for $j = 0, i = 1, 2$, $\omega_{ji} = l_i$.
(7) **Total amount sent (auxiliary variable)** $M_i(t) = y_i(t) + \sum_{j \in V_i} v_{ij}(t)$. We assume that $M_i(t) \leq S_i(t)$ $\forall i$.

In [8], we also defined $z_i(t)$ called *excess demand* function which is difference between total demand and availability and expressed it in the form

$$z_i(t+1) = SAT(\xi_i(t))\xi_i(t) + C_2 \cdot g_i(t-\tau_i) + d_i(t). \quad (1)$$

Here $Sat(f) = 1$ if $f \geq 0$ and $Sat(f) = 0$ if $f \leq 0$ and and

$$\xi_i(t) = -z_i(t) + B^i \cdot w_i(t-\tau_i) + C_1 \cdot g_i(t-\tau_i) \quad (2)$$

with $B^i = [1, \ldots, 1]$ is a row vector of length m_i, $C_1 = [1, 0, 0]$, $C_2 = [0, 1, 1]$, and

$$g_i(t-\tau_i) = \left[\begin{array}{c} \sum_{j \in V_i} u_{ji}(t-\tau_i-1) \\ \sum_{j \in V_i} u_{ji}(t-\tau_i) \\ \sum_{j \in U_i} v_{ij}(t-\tau_i-\omega_{ij}) \end{array} \right] \quad (3)$$

where

$$\tau_i = max_{j \in U_i} \{\tau_j + \omega_{ji}\} \quad (4)$$

and $\tau_0 = 0$. For each DC i, we also defined a function $\eta_i(t)$ by the following iteration.

$$\eta_i(t+1) = \eta_i(t) + \sum_{j \in U_i} (u_{ij}(t-1-\tau_i) - v_{ji}(t-\tau_i)) \quad (5)$$

where $\eta_i(0) = 0$. We note that $\eta_i(t)$ measures the cumulative discrepancy between what a DC i orders and what it receives. For optimal performance of the system, we require that

$$\sum_t E(\eta_i(t)) = 0. \quad (6)$$

Finally we defined a summing variable, $J_i(t)$ by

$$\begin{aligned} J_i(t+1) &= J_i(t) + (h_i - s_i\xi_i(t) \\ &+ SAT(\xi_i(t))\xi_i(t) + h_i d_i(t) + \eta_i(t) \quad (7) \end{aligned}$$

with initial condition $J_i(0) = J_{i_0}$ and showed that our final optimization problem characterizing the policies of each DC i is given by

$$min_{w_i, u_{ji}: j \in U_i} E\{J_i(N)\} \quad (8)$$

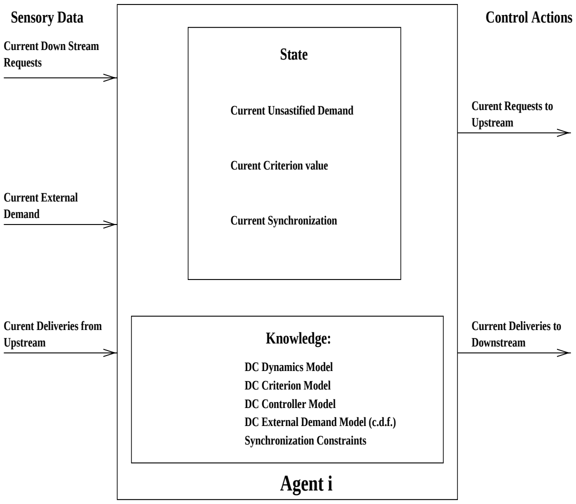

Figure 1: Agent Planner for DC i

subject to the dynamic constraints (1),(5) and (7) and their corresponding initial conditions plus the structural constraints that $\forall t$, $u_{ij}(t) \geq 0$, $v_{ij}(t) \geq 0$ and $\xi_i(t) \geq 0$.

2. Continualized Model

In this section we develop a continualized approximation of the agent model for a node in single product distribution network. Figure 1 shows the main elements of the agent that generates a planning policy for the DC i. The outputs of the continualization procedure are differential models. Any unexplained notation can be found in notation of [8].

For each DC i, let $r_i(t) = [z_i(t), \eta_i(t), J_i(t)]^T$. We can obtain an expression for the demand in terms of the state and total delivery and request decision variables for DC i, $r_i(t)$,$w_i(t)$, $\tilde{u}_i$, $g_i(t)$ and the request and delivery vector for the down stream DC j's. We can then construct the predictive state equations of DC i by simply advancing the time variable by τ_i. If we then replace in the resulting vector iteration all the terms that include $d_i(t)$ by its expression in terms of the state and control variables, the resulting state equation will be feedback function of the form of the form

$$\begin{aligned} & r_i(t+\tau_i+1) = \\ & \sum_{k \ inU_i \cup \{i\}} \sum_{j=0}^{\tau_i} \tilde{A}_{ik}(t) \cdot r_k(t+j)) + \tilde{B}_i(t) \cdot w_i(t) - \\ & \tilde{C}_i(t) \cdot \tilde{u}_i(t) + \tilde{G}_i(t) \cdot g_i(t) + \\ & \sum_{s=1}^{\tau_i+1} \tilde{D}_{is}(t+s) \cdot d_i(t+s) \quad (9) \end{aligned}$$

for appropriate matrices $\tilde{A}$, $\tilde{D}$, $\tilde{B}_i$, $\tilde{C}_i$, and $\tilde{G}_i$ where

$$\tilde{u}_i(t) = \left[\begin{array}{c} \tilde{u}_i^1(t) \\ \vdots \\ \tilde{u}_i^{m_i} i(t) \end{array} \right] = \left[\begin{array}{c} u_{j_1 i}(t) \\ \vdots \\ u_{m_i i}(t) \end{array} \right] \quad (10)$$

$$w_i(t) = \left[\begin{array}{c} w_i^1(t) \\ \vdots \\ w_i^{m_i}(t) \end{array} \right] = \left[\begin{array}{c} v_{ij_1}(t) \\ \vdots \\ v_{im_i}(t) \end{array} \right]. \quad (11)$$

Due to lack of space, we shall not derive the matrices $\tilde{A}$, $\tilde{D}$, $\tilde{B}_i$, $\tilde{C}_i$, and $\tilde{G}_i$. Moreover our main concern in this paper is how to continualize equation of the form (9) rather than the exact nature of the matrices $\tilde{A}$, $\tilde{D}$, $\tilde{B}_i$, $\tilde{C}_i$, and $\tilde{G}_i$ which, in general, depend upon the constraints of a particular problem.

The next step in the continualization process is to construct a feasible symbolic form of decision policy as a function of the state and demand. Because of the quasi-linearity of (9) and the form of the conservation principles in supply chain problems, we can express our generic feasible policy as a quasi-linear feedback of the form

$$\begin{aligned} w_i(t) &= \sum_{k=0}^{\tau_i - 1} \Theta_{ik}^{wr} \cdot \hat{r}_i(t+k) + \\ & \sum_{s=0}^{\tau_i} \Theta_{is}^{wd} \cdot \hat{d}_i(t+s) + \Theta_i^{wg} \cdot \hat{g}_i(t+1)) \\ \tilde{u}_i(t) &= \sum_{k=0}^{\tau_i - 1} \Theta_{ik}^{ur} \cdot \hat{r}_i(t+k) + \\ & \sum_{s=0}^{\tau_i} \Theta_{is}^{ud} \cdot \hat{d}_i(t+s) + \Theta_i^{ug} \cdot \hat{g}_i(t+1)). \end{aligned} \quad (12)$$

In (12), the ˆ on top of a variable denotes expectation conditioned on the data up to time t. Similarly, from (9) we obtain

$$\begin{aligned} & \hat{r}(t+\tau_i+1) = \\ & \sum_{k \in U_i \cup \{i\}} \sum_{j=0}^{\tau_i} \tilde{A}_{kj}(t) \cdot \hat{r}_k(t+j) + \tilde{B}_i(t) \cdot w_i(t) - \\ & \tilde{C}(t) \cdot \tilde{u}_i(t) + \tilde{G}_i(t) \cdot g_i(t) + \\ & + \sum_{s=0}^{\tau_i+1} \tilde{D}_{is}(t+s) \cdot \int_0^{\infty} d_i \cdot dF_i^{t+s}(d_i). \end{aligned} \quad (13)$$

If we then use the optimum coefficient functions in (12) and replace the resulting expressions for the decision variables $w_i(t)$ and $\tilde{u}_i(t)$ in (13), we will obtain an expression for the optimum state trajectory for the DC i in terms of the external demand and the request and delivery trajectories. Specifically, in (12), the 'coefficient' functions are to be determined so as to satisfy the optimality in (8), namely,

$$min_{\Theta_{ik}^{lm}} E\{r_i(N-\tau_i)\} \quad (14)$$

subject to (12) and (13) and their corresponding initial conditions plus the structural constraints that $u_{ji}(t) \geq 0$, $v_{ij}(t) \geq 0$, and $\xi_i(t) \geq 0$ for all t.

Before we continue to the next step of our continualization procedure, we pause to remark on a technical point of importance. We note that the coefficient functions in both (12) and (13) are piecewise twice differentiable (C^2) functions with only isolated point discontinuities that are caused by the SAT function. A version of the Smoothing Lemma ([4], pg. 249) establishes the existence of twice-differentiable models arbitrarily 'close' to a given piecewise C^2 model with finitely many isolated discontinuities where close here means in the strong norm sense. Therefore, we can assume that (9), (12) are the C^2 versions of the model. When continualization is mechanized in a computer procedure, it must be provided with a robust heuristic to 'round-up' these point discontinuities.

Plugging (12) into (13), we obtain an expression of predicted behavior dynamics for DC i. For simplicity of the calculations that follow, we will express this equation in 'state space notation' as

$$\begin{aligned} \hat{\mathbf{X}}^i(t+1) &= \mathbf{F}^i(t, \Theta^i(t)) \cdot \hat{\mathbf{X}}^i(t) + \\ & \mathbf{H}^i(t, \Theta^i(t)) \cdot \mathbf{G}^i(t) + \\ & \mathbf{E}^i(t, \Theta^i(t)) \cdot \hat{\mathbf{D}}^i(t) \end{aligned} \quad (15)$$

where $\hat{\mathbf{X}}^i(t) = [\hat{r}_i(t), \ldots, \hat{r}_i(t+\tau_i)]^T$, $\mathbf{G}^i(t)$ is the vector of requests and deliveries to DC i, and $\hat{\mathbf{D}}^i(t) = [\hat{d}_i(t), \ldots, \hat{d}_i(t+\tau_i)]^T$ is the vector of external demand forecasts between the current time t and the time $t + \tau_i$.

The state evolution of each DC i is a Markov process driven by external demand and the decision policy (12). Equation (15) shows how the conditional expectation of this process evolves in time.

The next step in the continualization process involves writing the necessary conditions of optimality in problem (14) for the coefficient functions as a time iteration. For this problem, this is particularly easy because the Hamiltonian is of the form

$$\begin{aligned} & 0 = \Omega^i(t, \Theta^i(t), \mathbf{X}^i(t), \mathbf{P}^i(t+1)) = \quad (16) \\ & [\mathbf{P}^i(t+1)]^T \cdot [\mathbf{F}^i(t, \Theta^i(t)) \cdot \hat{\mathbf{X}}^i(t) + \\ & \mathbf{H}^i(t, \Theta^i(t)) \cdot \mathbf{G}^i(t) + \mathbf{E}^i(t, \Theta^i(t)) \cdot \hat{\mathbf{D}}^i(t)] = 0 \end{aligned}$$

where $\hat{\mathbf{P}}^i(t) = [\hat{r}_i(t), \ldots, \hat{r}(t+\tau_i)]$, is a vector multiplier. Thus the optimality condition takes the form

$$\frac{\partial \Omega^i(t, \Theta^i(t), \mathbf{X}^i(t), \mathbf{P}^i(t+1))}{\partial \Theta^i} = [\mathbf{P}^i(t+1)]^T \cdot$$
$$\frac{\partial}{\partial \Theta^i}[\mathbf{F}^i(t, \Theta^i(t)) \cdot \hat{\mathbf{X}}^i(t) +$$
$$\mathbf{H}^i(t, \Theta^i(t)) \cdot \mathbf{G}^i(t) + \mathbf{E}^i(t, \Theta^i(t)) \cdot \hat{\mathbf{D}}^i(t)]$$
$$= 0 \tag{17}$$

For simplicity, let us define the function $\Xi(\Theta^i(t), \hat{\mathbf{X}}^i(t), \mathbf{G}^i(t), \hat{\mathbf{D}}^i(t))$ as follows.

$$\Xi(\Theta^i(t), \hat{\mathbf{X}}^i(t), \mathbf{G}^i(t), \hat{\mathbf{D}}^i(t)) =$$
$$\frac{\partial}{\partial \Theta^i}[\mathbf{F}^i(t, \Theta^i(t)) \cdot \hat{\mathbf{X}}^i(t) +$$
$$\mathbf{H}^i(t, \Theta^i(t)) \cdot \mathbf{G}^i(t) + \mathbf{E}^i(t, \Theta^i(t)) \cdot \hat{\mathbf{D}}^i(t)] \tag{18}$$

Then the Stochastic Approximation (SA) iteration for computing an approximating sequence that satisfies (17) is given by

$$\Theta^i(t+1) = \tag{19}$$
$$\Theta^i(t) + \epsilon^i(t) \cdot \Xi(\mathbf{\Theta^i(t)}, \mathbf{X^i(t)}, \mathbf{G^i(t)}, \hat{\mathbf{D}}^{\mathbf{i}}(\mathbf{t}))$$

where $\epsilon^i(t)$ is a time convergence parameter sequence such that

$$\epsilon^i(t) \to 0 \text{ as } t \to T, \sum_t \epsilon^i(t) \to M^i \text{ (large)}$$

We note that in the SA iteration (19), the convergence scalar parameter has replaced the multiplier function $\mathbf{P}^i$in (17) and hence no backward recursion is needed. Also, the convergence of this recursion can be established from the smoothness of the construction in (15).

Now equation (15) can also be written in the standard SA form as follows.

$$\hat{\mathbf{X}}^i(t+1) - \hat{\mathbf{X}}^i(t) =$$
$$(\mathbf{F}^i(t, \Theta^i(t)) - I) \cdot \hat{\mathbf{X}}^i(t) +$$
$$\mathbf{H}^i(t, \Theta^i(t)) \cdot \mathbf{G}^i(t) + \mathbf{E}^i(t, \Theta^i(t)) \cdot \hat{\mathbf{D}}^i(t). \tag{20}$$

Next we continualize (19) and (20). Let $\tilde{\Theta}^i(\sigma)$ and $\tilde{\mathbf{X}}^i(\sigma)$ be continuous processes in the continuous 'time' variable σ which are defined as follows.

$$\tilde{\Theta}^i(\sigma) =$$
$$\Xi_i(\tilde{\Theta}^i(\sigma), \tilde{\mathbf{X}}^i(\sigma), \tilde{\mathbf{G}}^i(\sigma), \tilde{\hat{\mathbf{D}}}^i(\sigma)) + \mathbf{B}_0^i(\sigma) \tag{21}$$

and

$$\begin{aligned} \tilde{\mathbf{X}}^i(\sigma) &= \mathbf{F}^i(\sigma, \tilde{\Theta}^i(\sigma) - I) \cdot \tilde{x}^i(\sigma) + \\ &\quad \mathbf{H}^i(t, \tilde{\Theta}^i(\sigma)) \cdot \tilde{\mathbf{G}}^i(\sigma) + \\ &\quad \mathbf{E}^i(\sigma, \tilde{\Theta}^i(\sigma)) \cdot \tilde{\hat{\mathbf{D}}}^i(\sigma)) + \mathbf{B}_1^i \end{aligned} \tag{22}$$

The vector functions, $\mathbf{B}_0^i, \mathbf{B}_1^i$, are convergence controls that tend to zero as $\sigma \to N$. The 'continuous-time' vector variables in (21) are related to the 'discrete-time' by the following linear interpolation schema.

$$\begin{aligned} \tilde{\Theta}^i(t) &= \Theta^i(t) \ \forall t \\ \tilde{\Theta}^i(\sigma) &= \frac{(t+1-\sigma)}{\epsilon(t)} \cdot \Theta^i(t) + \\ &\quad \frac{(\sigma - t)}{\epsilon(t)} \cdot \Theta^i(t+1) \\ &\quad \forall \sigma \in (t, t+1) \end{aligned} \tag{23}$$

$$\begin{aligned} \tilde{\mathbf{X}}^i(t) &= \hat{\mathbf{X}}^i(t) \ \forall t \\ \tilde{\mathbf{X}}^i(\sigma) &= \frac{(t+1-\sigma)}{\epsilon(t)} \cdot \hat{\mathbf{X}}^i(t) \\ &\quad \frac{(\sigma - t)}{\epsilon(t)} \cdot \hat{\mathbf{X}}^i(t+1) \\ &\quad \forall \sigma \in (t, t+1) \end{aligned} \tag{24}$$

Similar formulas hold for $\tilde{\hat{\mathbf{D}}}^i(\sigma)$ and $\tilde{\mathbf{G}}^i(\sigma)$. The following continuous time optimization problem has a solution that approximates the solution (14) up to order $o(\epsilon^2(N))$.

$$min_{\theta_{ik}^{lm}} \tilde{\mathbf{X}}_n^i(N - \tau_i) \tag{25}$$

subject to the constrains (23),(24), ... and their corresponding initial conditions plus the usual positivity constraints.

The last step of our continualization procedure is to transform problem (25) into a convex variational problem that generates the same state and parameter trajectory as (25). We note that the proof of the stated distance between solutions to problem (25) and the discrete solutions to problem (14) is not difficult. It is a rather lengthy and technical exercise that involves comparing Euler discretization sequences of (23), (24), ... with corresponding iterations of (19) and (20) and an application of the Arzela-Ascoli theorem for uniform convergence.

Let $\mathbf{Y}^i(\sigma) = [\tilde{\Theta}^i(\sigma), \tilde{\mathbf{X}}^i(\sigma)]$ and $\mathbf{f}^i(\mathbf{Y}^i(\sigma), \sigma)$ be the following function.

$$\mathbf{f}^i(\mathbf{Y}^i(\sigma), \sigma) = \tag{26}$$
$$\begin{bmatrix} \Xi_i(\tilde{\Theta}^i(\sigma), \tilde{\mathbf{X}}^i(\sigma), \tilde{\mathbf{G}}^i(\sigma), \tilde{\hat{\mathbf{D}}}^i(\sigma)) + \mathbf{B}_0^i(\sigma) \\ (\mathbf{F}^i(\sigma, \tilde{\Theta}^i(\sigma)_I) \cdot \tilde{x}^i(\sigma) + \mathbf{H}^i(t, \tilde{\Theta}^i(\sigma)) \cdot \tilde{\mathbf{G}}^i(\sigma) + \\ \mathbf{E}^i(\sigma, \tilde{\Theta}^i(\sigma)) \cdot \tilde{\hat{\mathbf{D}}}^i(\sigma)) + \mathbf{B}_1^i) \end{bmatrix}$$

Then we can write (21) as the following vector field equation.

$$\dot{\mathbf{Y}}^i(\sigma) = \mathbf{f}^i(\mathbf{Y}^i(\sigma), \sigma). \tag{27}$$

By our construction via the Smoothing Lemma [4] $\mathbf{f}^i(\mathbf{Y}^i(\sigma),\sigma)$ is twice continuously differentiable in both arguments and hence we can differentiate both sides of (27) with respect to σ to obtain the following spray equation associated with (27).

$$\begin{aligned}\ddot{\mathbf{Y}}^i(\sigma) &= \frac{\partial \mathbf{f}^i(\mathbf{Y}^i(\sigma),\sigma)}{\partial \mathbf{Y}^i}\cdot\dot{\mathbf{Y}}^i(\sigma)+\frac{\partial \mathbf{f}^i(\mathbf{Y}^i(\sigma),\sigma)}{\partial\sigma}\\ &= \mathbf{F}^i(\mathbf{Y}^i(\sigma),\dot{\mathbf{Y}}^i(\sigma),\sigma). \end{aligned}\qquad (28)$$

We want to find a function $L^i(\mathbf{Y}^i(\sigma),\dot{\mathbf{Y}}^i(\sigma),\sigma)$ such that a solution of the problem

$$min_{\mathbf{Y}^i}\int_0^T L^i(\mathbf{Y}^i(\xi),\dot{\mathbf{Y}}^i(\xi),\xi)\cdot d\xi + Y^i_{2n}(T) \qquad (29)$$

subject our initial conditions and positivity constraints is also a solution of problem (25). Specifically, we seek a function L^i such that (28) are necessary conditions for optimality in problem (29). In our previous papers, we refer to this computation as the *inverse Lagrangian procedure.* Now the necessary conditions for optimality of (29) are given by the Euler-Lagrange Equations associated with L^i, namely,

$$\begin{aligned}&\frac{d}{d\sigma}\frac{\partial L^i(\mathbf{Y}^i(\sigma),\dot{\mathbf{Y}}^i(\sigma),\sigma)}{\partial \dot{\mathbf{Y}}^i_j}-\frac{\partial L^i(\mathbf{Y}^i(\sigma),\dot{\mathbf{Y}}^i(\sigma),\sigma)}{\partial \mathbf{Y}^i_j}\\ &=0\end{aligned}\qquad (30)$$

for $j=1,\ldots,n_i$. Using (28) in (30) we can write (30) as follows.

$$\sum_{j=1}^{n_i}[L^i_{\dot{\mathbf{Y}}^i_m\mathbf{Y}^i_j}\cdot\dot{\mathbf{Y}}^i_j+L^i_{\dot{\mathbf{Y}}^i_m\dot{\mathbf{Y}}^i_j}\cdot\mathbf{F}^i_j]+L^i_{\dot{\mathbf{Y}}^i_m\sigma}-L^i_{\mathbf{Y}^i_m}=0 \qquad (31)$$

for $m=1,\ldots,n_i$.

In (31), we have used the sub-index notation for partial derivatives. By our construction L is a smooth function. Therefore, in (31) we may differentiate with respect to $\dot{\mathbf{Y}}^i_k$. After some algebra, we obtain the following from (31). For all $m=1,\ldots,n_i$ and $k=1,\ldots,n_i$,

$$\begin{aligned}0 \;=\; &\sum_{j=1}^{n_i}2L^i_{\dot{\mathbf{Y}}^i_m\dot{\mathbf{Y}}^i_k\mathbf{Y}^i_j}\cdot\dot{\mathbf{Y}}^i_j+\\ &\sum_{j=1}^{n_i}2L^i_{\dot{\mathbf{Y}}^i_m\dot{\mathbf{Y}}^i_k\dot{\mathbf{Y}}^i_j}\cdot\mathbf{F}^i_j+\sum_{j=1}^{n_i}2L^i_{\dot{\mathbf{Y}}^i_m\dot{\mathbf{Y}}^i_j}\cdot\mathbf{F}^i_{j_{\mathbf{Y}^i_k}}+\\ &\sum_{j=1}^{n_i}2L^i_{\dot{\mathbf{Y}}^i_m\dot{\mathbf{Y}}^i_k\mathbf{Y}^i_j}\cdot\mathbf{F}^i_{j_{\mathbf{Y}^i_m}}+2L^i_{\dot{\mathbf{Y}}^i_m\dot{\mathbf{Y}}^i_k\sigma}\end{aligned}\qquad (32)$$

Now we define the symmetric matrix function $\mathbf{\Psi^i}(\mathbf{Y^i},\dot{\mathbf{Y}}^\mathbf{i},\sigma)$ to be the Hessian of L^i. That is,

$$\mathbf{\Psi^i_{km}}(\mathbf{Y^i},\dot{\mathbf{Y}}^\mathbf{i},\sigma)=\mathbf{L^i_{\dot{Y}^i_k\dot{Y}^i_m}}(\mathbf{Y^i},\dot{\mathbf{Y}}^\mathbf{i},\sigma) \qquad (33)$$

We also have that the total derivative of $\mathbf{\Psi^i}(\mathbf{Y^i},\dot{\mathbf{Y}}^\mathbf{i},\sigma)$ with respect to σ is given by

$$\frac{d\mathbf{\Psi^i}}{d\sigma}=\mathbf{\Psi^i_\sigma}+\mathbf{\Psi^i_{Y^i}}\cdot\mathbf{Y^i}+\mathbf{\Psi^i_{\dot{Y}^i}}\cdot\mathbf{F^i}. \qquad (34)$$

For the Hessian $\left[L^i_{\dot{\mathbf{Y}}^i_k\dot{\mathbf{Y}}^i_m}\right]$ of L^i the equations in (32) define a quasi-linear hyperbolic partial differential equation. Indeed, using (34) and (33) in (32), we obtain the following differential equation for $\mathbf{\Psi^i}$ along the characteristics associated with the equation.

$$\frac{d\mathbf{\Psi^i}}{d\sigma}=-\frac{1}{2}\mathbf{\Psi^i}\cdot(\mathbf{F^i_{\dot{Y}^i}})^\mathbf{T}+-\frac{\mathbf{1}}{\mathbf{2}}\mathbf{F^i_{\dot{Y}^i}}\cdot\mathbf{\Psi^i}. \qquad (35)$$

Note that (35) is just a a linear Lyapunov equation!

A positive, convex Lagrangian L^i_0 corresponding to (35) is given by the following quadrature.

$$L^i_0(\mathbf{Y}^i,\dot{\mathbf{Y}}^i,\sigma)=\frac{1}{2}(\dot{\mathbf{Y}}^i)^T\cdot\mathbf{\Psi^i}\cdot\dot{\mathbf{Y}}^\mathbf{i} \qquad (36)$$

From this Lagrangian an equivalent Lagrangian L^i_1 can be defined in which the dependency on the time parameter is made implicit by the following equation.

$$L^i_1(\mathbf{Y}^i,\dot{\mathbf{Y}}^i)=\frac{1}{2}[(\dot{\mathbf{Y}}^i)^T 1]\cdot\begin{bmatrix}\mathbf{\Psi^i} & 0\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}\dot{\mathbf{Y}}^i\\ 1\end{bmatrix}. \qquad (37)$$

Finally, from (36) we can obtain a Lagrangian that is *homogeneous* in the rate variables. This is particularly important because the convergence in the algorithm called Hybrid Dynamic Programming (HDP) [7] that our agents use to solve the optimization problem in (29) is accelerated for an homogeneous Lagrangian.

We conclude this paper by producing yet another equivalent Lagrangian L^i_2 which a homogenized version of $L^i_1(\mathbf{Y}^i,\dot{\mathbf{Y}}^i)$. Let $f(\sigma)$ be a positive twice-differentiable function. Then the Lagrangian

$$L^i_2([f,\mathbf{Y}^i],[\dot{f},\dot{\mathbf{Y}}^i])=\sqrt{L^i_1(\mathbf{Y}^i,\frac{\dot{\mathbf{Y}}^i}{\dot{f}})\cdot\dot{f}} \qquad (38)$$

is positive homogeneous of degree one in the rate. That is, f or any positive real number λ,

$$L^i_2([f,\mathbf{Y}^i],[\lambda\dot{f},\lambda\dot{\mathbf{Y}}^i])=\lambda L^i_2([f,\mathbf{Y}^i],[\dot{f},\dot{\mathbf{Y}}^i]).$$

Working with homogeneous of degree 1 Lagrangians greatly simplifies the HDP algorithm [7] that

MACHA agent use to computing an optimal solution of problem (39) given below.

$$min_{\mathbf{Y}^i} \int_0^T L_2^i([f(\xi), (\mathbf{Y}^i(\xi)], [\dot{f}(\xi), \dot{\mathbf{Y}}^i(\xi)]) \cdot d\xi + Y_{2n}^i(T) \tag{39}$$

subject to our initial conditions and positivity constraints. The optimization problem in (39) is our desired continualized formulation.

3. Output Procedure

In this section, we prescribe how to obtain an arbitrarily close approximation to the minimal stocking plan from the locally unique solution of (39). This is summarized below as *output procedure*. This is the procedure our MAHCA agents use to generate at each interval the 'output values' as a function of the agents internal continualized variables. Although the multi-echelon planner of this paper need not be implemented in real time, our construction would allow such implementation.

Output Procedure

Let $\mathbf{Y}_*^i(\sigma)$ for $\sigma \in [0, T]$ be the solution to optimization problem (39). Then, we construct the solution to the original problem for each DC i and at each discrete time t for $t = 0, 1, \ldots T$ as follows.
1) First get the discrete time coefficient functions $\Theta^i(t)$ and the state estimate $\hat{\mathbf{X}}^i(t)$. Then from (23) find

$$\mathbf{Y}_*^i(t) = \left[\begin{array}{c} \Theta_*^i(t) \\ \hat{\mathbf{X}}_*^i(t) \end{array} \right] \text{ for } t \in \{0, 1, \ldots, T\}. \tag{40}$$

2) Next using (12), compute feedback expressions for the decision variables $\tilde{w}_{i*}$ and $\tilde{u}_{i*}$.
3) Finally, using the results of step 2 in equation (13), we can compute the discrete state as a function of the decision variables, the downstream requests and the external demand. We may use the difference between the state estimate constructed from step 1 and the one constructed here, to generate an estimate of the error that may be used for improvement in the computed stock plan.

4. Conclusions

In this paper, we have outlined how to take a predicitive feedback functional equation 17), that arose out of the procedure to design a Multiple Agent Hybrid Control Architecture (MAHCA) network for the deployment of a reactive, agent based planner for a single product, multi-echelon inventory system described in [8], and continualize it so as to transform the problem to a nonlinear optimization problem of the form of computing an optimal solution of problem

$$min_{\mathbf{Y}^i} \int_0^T L_2^i([f(\xi), (\mathbf{Y}^i(\xi)], [\dot{f}(\xi), \dot{\mathbf{Y}}^i(\xi)]) \cdot d\xi + Y_{2n}^i(T) \tag{41}$$

subject to our initial conditions and positivity constraints where L_2^i is psitive homogeneous of degree 1. Equations of the form (41) are precisely the type of problems that can be computed by MAHCA agents via a hybrid dynamic programming algorithm, see [7, 6, 5].

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